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1994 J. Phys. A: Math. Gen. 27 295

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## Lattice magnetic walks

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Received 2 June 1993

**Abstract.** Sums of walks for charged particles (e.g. Hofstadter electrons) on a square lattice in the presence of a magnetic field are evaluated. Returning loops are systematically added to directed paths to obtain the unrestricted propagators. Expressions are obtained for special values of the magnetic flux-per-plaquette commensurate with the flux quantum. For commensurate and incommensurate values of the flux, the addition of small returning loops does not affect the general features found earlier for directed paths. Lattice Green functions are also obtained for staggered flux configurations encountered in models of high- $T_c$  superconductors.

### 1. Introduction

The problem of electrons confined to a lattice in the presence of a transverse magnetic field is a classic problem of mathematical physics with many applications in condensed-matter physics. The early works of Hofstadter [1], Wannier [2] and Azbel [3] focused on the spectral properties of non-interacting electrons as a function of the energy and the parameter  $\alpha = \phi/\phi_0$ , where  $\phi$  is the magnetic flux-per-plaquette and  $\phi_0 = hc/e$  is the flux quantum. For commensurate  $\alpha = p/q$  the spectrum has  $q$  subbands with special scaling properties as  $q$  grows larger, eventually turning into a Cantor set for irrational  $\alpha$ . Hence, the behaviour of the spectrum and the wavefunction is very sensitive to the exact value of  $\alpha$  and the energy  $E$  (the intricate structure of the bands as a function of  $\alpha$  and  $E$  has been dubbed the ‘butterfly’).

Lately the renewed interest in interacting electrons in two dimensions, in the context of the quantum Hall effect and various theories for high- $T_c$  superconducting materials for instance, has led to more recent investigations of this general problem. Over time, these works have yielded an extraordinary richness using a variety of techniques including path integrals [4], renormalization groups [5], sophisticated algebraic methods [6] closely related to ‘quantum groups’, and so on.

More recently the study of non-interacting electrons in a magnetic field has been approached from a new direction—that of ‘localized’ wavefunctions [7–10]. If the energy of the electron does not lie within a quasiband, its eigenfunction is not a bulk (extended) eigenstate. However, such a wavefunction may be localized at inhomogeneities, such as the edge of the lattice (i.e. surface gap states) or at isolated impurities in an otherwise ordered lattice. These states decay exponentially as one moves away from the inhomogeneity into the bulk. The effect of the magnetic field on this exponential decay (and particularly

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the associated 'localization length') has been studied in the so-called 'directed-paths approximation' [7, 10].

These calculations are based on the transfer-matrix approach (reviewed below) which is applicable only in this approximation. While it can be shown that this approximation yields a good description in the extremely localized limit in the absence of any field, the question is more delicate when the magnetic field is present. Indeed, to the next order, paths with small returning loops should be added. In the absence of a magnetic field, this inclusion leads to a simple renormalization of the 'localization length'. On the other hand, one already finds in the directed-paths approximation that in the presence of the field, the combined effects of lattice periodicity and the magnetic flux yield a complex interference pattern with a very sensitive dependence of the localization length upon the flux [10]. Therefore, the effects of including even small returning loops with their enclosed flux are unpredictable and potentially very drastic.

Including the returning loops, as explained below, becomes even more important if the energy to begin with is not extremely far from the band edge and the wavefunctions are consequently only moderately localized. We thus set out in this work to go beyond the directed-paths approximation and to calculate systematically the full Green functions which include the sum over all paths. These will be given as formal expressions which may be expanded systematically in various ways. Notable among these is the expansion in terms of the path length starting with the directed (the shortest) ones; it reveals that the generic features found in the directed-paths limit are not drastically changed by the small returning loops. In this paper we restrict ourselves to the square lattice, but using this approach, the sums for other lattices may be calculated as well. The expression we derive may also be useful to analyse properties of Hofstadter electrons within the band of bulk eigenstates.

Non-uniform flux configurations have been studied recently in the context of theories of strongly interacting electrons relevant to high- $T_c$  cuprates. The most important among them are the staggered flux configurations [11]. We consequently obtain here the lattice Green function for the square lattice with staggered flux as well.

The layout of this paper is as follows. In section 2 we review the model and the 'directed-paths-only' transfer-matrix approach. The heart of the paper is section 3, where we consider the addition of 'backward excursions' to the transfer-matrix method. Here, we find recursion relations and other expressions for the various quantities involved. In section 4, we examine some special cases thereof. Section 5 demonstrates how the full Green function can be obtained from the quantities derived in sections 3 and 4. Section 6 considers the staggered flux configuration, and section 7 contains a discussion of the results. We also include an appendix, listing some additional results.

## 2. The model and the approach

One can model non-interacting electrons confined to a square lattice with a perpendicularly applied magnetic field by the following tight-binding Hamiltonian:

$$H = \epsilon \sum_i a_i^\dagger a_i + V \sum_{\langle i,j \rangle} a_i^\dagger a_j e^{i\gamma_{ij}} \quad (2.1)$$

where  $\epsilon$  is the on-site energy and  $V$  measures the overlap of wavefunctions centred at neighbouring sites. The energy spectrum is the same as the one obtained using the 'Peierls substitution', which replaces  $\hbar\mathbf{k}$  with  $\mathbf{p} - e\mathbf{A}/c$  in the dispersion relation obtained from the

tight-binding approximation in the non-magnetic case. The phase  $\gamma_{ij} = -\gamma_{ji}$  denotes the Aharonov–Bohm phase an electron acquires as it moves from site  $i$  to site  $j$ ; physically, it corresponds to the line integral of the vector potential from  $i$  to  $j$ . Some freedom exists in the choice of gauge: to characterize a constant magnetic field, one selects the phases such that the directional sum of the phases  $\gamma_{ij}$  around each elementary plaquette is  $2\pi\phi/\phi_0$  where  $\phi$  is the magnetic flux-per-plaquette and  $\phi_0$  the flux quantum ( $hc/e$ ).

Many of the physically interesting quantities are contained in or are derivable from the Green function associated with equation (2.1). One can express the (real-space) Green function  $\mathcal{G}(i, f, E, \phi)$  between sites  $i$  and  $f$  for an electron of energy  $E$  as a sum over paths connecting the two sites

$$\mathcal{G}(i, f; E, \phi) = \sum_{\text{paths}} \prod_{\langle jk \rangle} \frac{V \exp(i\gamma_{jk})}{E - \epsilon} \tag{2.2}$$

where the product is over steps comprising a given path. Each path carries a weight  $v^L$ , where  $v = V/(E - \epsilon)$  and  $L$  is the length of the path. Since the total number of paths on a square lattice grows as  $4^L$ , the sum should converge provided  $v < \frac{1}{4}$ . The model has a phase transition from localized to extended states occurring at  $v = \frac{1}{4}$  [5].

In the strong localization regime ( $v \ll \frac{1}{4}$ ), the electronic wavefunction decays quite rapidly as one moves away from some initial position. In this case,  $\mathcal{G}(i, f; E, \phi)$  is dominated by the shortest paths—those ‘directly’ connecting the initial and final sites [12]. Note that on a lattice there exists in general more than one shortest path connecting two sites. The presence of a magnetic field gives rise to interference among various constituent paths. Since it is the interference phenomena that interests us and since the difference in the phases accumulated between two paths is proportional to the area they enclose, we find it convenient to study the geometry in which directed paths can enclose the largest possible areas. For this reason, we begin our study with paths directed along the diagonal of a square lattice; we will call the directed axis  $t$  and the transverse axis  $x$  (see figure 1). Summation of paths directed along a lattice axis (rather than the diagonal) is also possible but slightly less convenient.

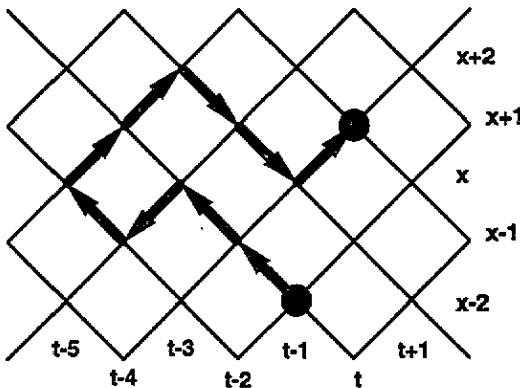


Figure 1. The diagonal lattice with  $t$  and  $x$  axes labelled and an example of a directed path with backward excursion.

One can employ the transfer-matrix approach to sum over directed paths [7]. If the vector  $|v_{t-1}\rangle$  represents a walk, or walkers, at column  $t - 1$ , then the matrix  $T(t, \phi)$  operating on

$|v_{t-1}\rangle$

$$|v_t\rangle = T(t, \phi)|v_{t-1}\rangle \tag{2.3}$$

provides all possible extensions of these walks by one forward step and weights these possibilities appropriately. Hence, consecutive operation by transfer matrices sums over all possible walks

$$|v_t\rangle = T(t, \phi)T(t - 1, \phi), \dots, T(1, \phi)|v_0\rangle. \tag{2.4}$$

Summing over paths weighted by different phase factors yields the desired interference effects.

The transfer matrix connecting column  $t - 1$  to column  $t$  is

$$T(t, \phi) = v \begin{bmatrix} 0 & e^{-it\phi\pi} & 0 & \dots & 0 & 0 & e^{it\phi\pi} \\ e^{it\phi\pi} & 0 & e^{-it\phi\pi} & \dots & 0 & 0 & 0 \\ 0 & e^{it\phi\pi} & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & e^{-it\phi\pi} & 0 \\ 0 & 0 & 0 & \dots & e^{it\phi\pi} & 0 & e^{-it\phi\pi} \\ e^{-it\phi\pi} & 0 & 0 & \dots & 0 & e^{it\phi\pi} & 0 \end{bmatrix} \tag{2.5}$$

where the bonds are viewed as being directed from lower to higher  $t$ . We exploit the gauge degree-of-freedom to furnish a transfer matrix that depends only on  $t$  (and not on  $x$ ). This choice is called the diagonal-staggered gauge [7]. As required, summing the phases as one proceeds clockwise around any elementary plaquette results in  $2\pi\phi$ ; we have set  $\phi_0 = hc/e = 1$  for convenience.

Note that in addition we have applied periodic boundary conditions along the  $x$  axis. However, since we approach the problem from the localization side, we may always take the lateral extent  $x_{\max}$  of the contributing paths to be smaller than the vertical size of the system  $L_{\perp}$ , and so the sum over paths will not be sensitive to the boundary conditions [13]. With periodic boundary conditions, the eigenvectors of all  $T$ 's are plane waves:

$$|v_k\rangle = \frac{1}{\sqrt{L_{\perp}}} \begin{bmatrix} e^{ik_{\perp}} \\ e^{i2k_{\perp}} \\ e^{i3k_{\perp}} \\ \vdots \\ e^{iL_{\perp}k_{\perp}} \end{bmatrix}. \tag{2.6}$$

The corresponding eigenvalues are  $\lambda_{k_{\perp}}(t, \phi) = 2v \cos(k_{\perp} - t\phi\pi)$ . The transverse momenta take on values  $k_{\perp} = 2\pi m/L_{\perp}$ , where  $L_{\perp}$  is the vertical size of the lattice and  $m = 0, 1, \dots, L_{\perp} - 1$ . The eigenvalues might also be called  $G_{k_{\perp}}^{\text{dir}}(t - 1, t, \phi)$  for they correspond to the Green function for directed walks (Fourier transformed in the transverse direction) joining columns  $t - 1$  and  $t$ .

Note that just as in the continuum model, one finds plane waves in the 'transverse' direction, leaving an effective one-dimensional problem (an harmonic oscillator in the continuum case). The reduction to a one-dimensional problem does not depend on the restriction to directed walks since the same transfer matrix also yields a 'backward' step from column  $t$  to  $t - 1$ . Henceforth, we need only consider eigenvalues, as all matrices of interest are simultaneously diagonalizable by the plane-wave eigenvectors.

In the presence of a magnetic field, the exponential decay of the electron wavefunction away from its central position is modulated by a very intricate interference pattern. In the strongly localized regime, many of its properties can be uncovered by investigating the product obtained from the directed paths only

$$G_{k_{\perp}}^{\text{dir}}(0, T_0, \phi) = \prod_{t=0}^{T_0-1} G_{k_{\perp}}^{\text{dir}}(t, t+1, \phi) = \prod_{t=1}^{T_0} \lambda_{k_{\perp}}(t, \phi) = \prod_{t=1}^{T_0} [2vC_t] \quad (2.7a)$$

where, for convenience, we have introduced the notation

$$C_t = \cos(k_{\perp} - t\phi\pi) \quad (2.7b)$$

suppressing the  $\phi$  and  $k_{\perp}$  dependences as most of the calculations considered herein involve a single  $\phi$  and a single  $k_{\perp}$ .  $G_{k_{\perp}}^{\text{dir}}(0, T_0, \phi)$  supplies the Green function for directed walks that begin at the origin and end in column  $T_0$ . The properties of this product have been the focus of a recent investigation [10]. The transition amplitude between two sites, say  $(0, 0)$  and  $(x_0, T_0)$ , may be obtained from them through:

$$G(0, 0; x_0, T_0) = \frac{1}{L_{\perp}} \sum_{k_{\perp}} G_{k_{\perp}}^{\text{dir}}(0, T_0; \phi) e^{-ik_{\perp}x_0}. \quad (2.8)$$

In the absence of any field, the directed Green function is simply a product of cosines:

$$G_{k_{\perp}}^{\text{dir}}(0, T_0, 0) = [2v \cos(k_{\perp})]^{T_0}. \quad (2.9)$$

When the applied flux-per-plaquette is a rational multiple of the flux quantum (i.e.  $\phi/\phi_0 = p/q$  with  $p$  and  $q$  relatively prime integers), the interference-induced modulation is periodic; in fact, the pattern on sites of a *superlattice* with lattice constant  $q$  times larger than the original lattice precisely mimics the simpler decay pattern found in the absence of an applied field [7–10]. If we restrict our attention to sites on the superlattice in the commensurate case, then

$$G_{k_{\perp}}^{\text{dir}}(0, T_0, p/q) = \left[ \prod_{t=1}^q 2vC_t \right]^{T_0/q}. \quad (2.10)$$

One can then use the following property

$$\prod_{t=1}^q C_t = \begin{cases} 2^{-q+1} \cos(qk_{\perp}) & \text{if } q \text{ is odd} \\ 2^{-q+1} \sin(qk_{\perp}) & \text{if } q \text{ is even} \end{cases} \quad (2.11)$$

to see the relation between the commensurate case on the superlattice and the non-magnetic case. Note that in the commensurate case,  $v^{-T_0} G_{k_{\perp}=0}^{\text{dir}}(0, T_0, p/q) \propto 2^{T_0/q}$ .

The work of Fishman *et al* [10] addresses the behaviour of this quantity in the incommensurate case (i.e. when  $q \rightarrow \infty$ ). In this case, the structure becomes aperiodic. For generic irrational, it has been found that  $\ln |v^{-T_0} G^{\text{dir}}|$  increases as  $[\ln(T_0)]^2$ ; while for an algebraic irrational  $\ln |v^{-T_0} G^{\text{dir}}|$  increases as  $\ln(T_0)$ .

### 3. Allowing backward excursions

The main motivation for the present work is to consider the consequences of lifting the 'directed-paths-only' restriction from the previous analysis, thereby broadening its scope beyond the strongly localized regime. We have opted to pursue the transfer-matrix method; however, this choice requires reconciling this approach, so ideally suited to directed paths, with our current concern of including returning loops. We explain first how the formal expressions can be obtained in an elegant and minimal way; we then proceed to a more detailed derivation which starts from the directed paths and adds systematically longer and longer returning loops. The expressions gained by this latter method will form the basis for our conclusions on the effects of adding small loops to fully directed paths. With these ends in mind, let us begin our investigation of 'directed paths with backward excursions', which we define next.

Let  $\tilde{\lambda}_{k_{\perp}}(t, \phi)$  be the eigenvalue of a transfer matrix  $\tilde{T}(t, \phi)$  which allows any amount of backward excursion, that is, any number of loops of any length starting from  $t - 1$  and ending at  $t$ —provided only the last step bridges columns  $t - 1$  and  $t$  (see, for example, figure 1). The eigenvalue  $\tilde{\lambda}_{k_{\perp}}(t, \phi)$  might also be called  $G_{k_{\perp}}^{\text{bc}}(t - 1, t, \phi)$  for it denotes the Green function linking columns  $t - 1$  and  $t$  that admits 'backward excursions'. The full, unrestricted Green function can then be calculated from  $G_{k_{\perp}}^{\text{bc}}(s, t, \phi)$  as will be shown in section 5.

In view of the fact that all of the terms comprising  $\tilde{\lambda}_{k_{\perp}}(t, \phi)$  contain a factor  $2vC_t$  corresponding to the last step, it is convenient to define  $\hat{\lambda}_{k_{\perp}}(t, \phi) = \tilde{\lambda}_{k_{\perp}}(t, \phi)/2vC_t$  with this last piece stripped off. Note that the walks contributing to  $\hat{\lambda}_{k_{\perp}}(t, \phi)$  begin and end in column  $t - 1$ , never reaching column  $t$ . Again we could use the Green function notation  $\hat{\lambda}_{k_{\perp}}(t, \phi) = G_{k_{\perp}}^{\text{bc}}(t - 1, t - 1; \phi)$  which is more descriptive but a bit more cumbersome, and so we will put off its use until section 5.

We have found that  $\hat{\lambda}_{k_{\perp}}(t, \phi)$  obeys the recursion relation:

$$\hat{\lambda}_{k_{\perp}}(t, \phi) = 1 + [2vC_{t-1}]^2 \hat{\lambda}_{k_{\perp}}(t - 1, \phi) + \{[2vC_{t-1}]^2 \hat{\lambda}_{k_{\perp}}(t - 1, \phi)\}^2 + \dots \quad (3.1a)$$

This relation can be understood in the following way: in the expansion, the first term on the right-hand side (the 1) indicates the option of making no backward steps. In the second term, one of the  $2vC_{t-1}$ s denotes a step back to column  $t - 2$ , then the  $\hat{\lambda}_{k_{\perp}}(t - 1, \phi)$  designates any amount of backward excursion beyond  $t - 2$  which eventually returns to column  $t - 2$  (including again the option of no excursion at all) and the second  $2vC_{t-1}$  returns the walker to column  $t - 1$ . Finally this whole process might be repeated any number of times as indicated by the expansion. The expansion is readily summed (at least formally) as a geometric series.

$$\hat{\lambda}_{k_{\perp}}(t, \phi) = \{1 - 4v^2 C_{t-1}^2 \hat{\lambda}_{k_{\perp}}(t - 1, \phi)\}^{-1}. \quad (3.1b)$$

Next, consecutive  $\hat{\lambda}$ 's [ $\hat{\lambda}_{k_{\perp}}(t - 1, \phi)$ ,  $\hat{\lambda}_{k_{\perp}}(t - 2, \phi)$ , ...] can be substituted into equation (3.1b) to arrive at the following continued-fraction expression:

$$\hat{\lambda}_{k_{\perp}}(t, \phi) = \frac{1}{1 - \frac{4v^2 C_{t-1}^2}{1 - \frac{4v^2 C_{t-2}^2}{1 - \frac{4v^2 C_{t-3}^2}{\text{etc}}}}}. \quad (3.2)$$

Terminating this continued fraction at the  $C_{t-s}^2$  term and re-expressing  $\hat{\lambda}$  as a ratio of polynomials, one finds that the numerator and denominator have essentially the same form except that the terms containing  $C_{t-1}$  are absent in the numerator. More specifically, the procedure leads to

$$\hat{\lambda}_{k_{\perp}}(t, \phi) = \frac{D_{k_{\perp}}(t, \phi; 2, s)}{D_{k_{\perp}}(t, \phi; 1, s)} + O(v^{2s+2}) \tag{3.3a}$$

where

$$\begin{aligned} D_{k_{\perp}}(t, \phi; r, s) = & 1 - 4v^2 \sum_{j=r}^s C_{t-j}^2 + 16v^4 \sum_{j=r}^s \sum_{k=j+2}^s C_{t-j}^2 C_{t-k}^2 \\ & - 64v^6 \sum_{j=r}^s \sum_{k=j+2}^s \sum_{l=k+2}^s C_{t-j}^2 C_{t-k}^2 C_{t-l}^2 + \dots \end{aligned} \tag{3.3b}$$

The problem has thus been reduced to that of understanding the properties of this particular function  $D_{k_{\perp}}(t, \phi; r, s)$ . Some of its more obvious and useful properties are:

$$D_{k_{\perp}}(t, \phi; r, s) = D_{k_{\perp}}(t+n, \phi; r+n, s+n) \tag{3.4a}$$

$$D_{k_{\perp}}(t, \phi; r, s) = D_{k_{\perp}}(t, \phi; r+1, s) - 4v^2 C_{t-r}^2 D_{k_{\perp}}(t, \phi; r+2, s) \tag{3.4b}$$

$$D_{k_{\perp}}(t, \phi; r, s) = D_{k_{\perp}}(t, \phi; r, s-1) - 4v^2 C_{t-s}^2 D_{k_{\perp}}(t, \phi; r, s-2). \tag{3.4c}$$

Much is encoded in equation (3.3). However, since our motivation is to extend the analysis of fully directed walks to include returning loops, we present a methodical derivation which does just that. Along the way useful expansions for  $\hat{\lambda}_{k_{\perp}}(t, \phi)$  are obtained.

Returning then to the directed paths, a first step towards including returning loops would permit, in addition to one forward step connecting columns  $t-1$  and  $t$ , one backward step followed by two forward steps. The modified eigenvalues  $\lambda_{k_{\perp}}^m(t, \phi)$  become

$$\lambda_{k_{\perp}}^m(t, \phi) = \lambda_{k_{\perp}}(t, \phi) + [\lambda_{k_{\perp}}(t-1, \phi)]^2 \lambda_{k_{\perp}}(t, \phi). \tag{3.5}$$

Let us analyse the effect of this addition on the Green function for one plaquette on the superlattice in the case of commensurate flux. As in equation (2.7a) for directed paths, the Green function is a product of these eigenvalues, and this product can be separated into two products as follows

$$G_{k_{\perp}}^m(0, q, p/q) = \left[ \prod_{t=1}^q 2vC_t \right] \times \left[ \prod_{t=1}^q (1 + 4v^2 C_{t-1}^2) \right]. \tag{3.6a}$$

The first product is  $G_{k_{\perp}}^{\text{dir}}(0, q, p/q)$  which we considered in equations (2.9) and (2.10); the second product is also readily handled; together they yield

$$G_{k_{\perp}}^m(0, q, p/q) = G_{k_{\perp}}^{\text{dir}}(0, q, p/q) \times (1 - \alpha)^{-2q} [1 - (-1)^q 2\alpha^q \cos(k_{\perp}q) + \alpha^{2q}] \tag{3.6b}$$

where

$$\alpha = \frac{1 + 2v^2 - \sqrt{1 + 4v^2}}{2v^2} \approx v^2 - 2v^4. \tag{3.6c}$$



From this expression, one can see that the main effect is a renormalization of  $v$  without any dependence on  $\phi$ . The only structural change arises from the  $\cos(k_{\perp}q)$  term; however, it is of the order  $v^{2q}$  times smaller than the leading behaviour. Next, let us consider the inclusion of longer backward excursions.

As one moves further out of the strongly localized regime, one should account for paths with increasingly longer backward excursions, culminating in  $\tilde{\lambda}_{k_{\perp}}(t, \phi)$  which consists of contributions of all lengths.  $\tilde{\lambda}_{k_{\perp}}(t, \phi)$  can be written as

$$\tilde{\lambda}_{k_{\perp}}(t, \phi) = \sum_{j=0}^{\infty} \tilde{\lambda}_{k_{\perp}}^{(j)}(t, \phi) \tag{3.7}$$

where  $\tilde{\lambda}_{k_{\perp}}^{(j)}(t, \phi)$  designates the contribution of length  $(2j + 1)$  which therefore carries a weight  $(2v)^{2j+1}$ . The first three are:

$$\tilde{\lambda}_{k_{\perp}}^{(0)}(t, \phi) = (2v)C_t \tag{3.8a}$$

$$\tilde{\lambda}_{k_{\perp}}^{(1)}(t, \phi) = (2v)^3 C_t C_{t-1}^2 \tag{3.8b}$$

$$\tilde{\lambda}_{k_{\perp}}^{(2)}(t, \phi) = (2v)^5 C_t [C_{t-1}^4 + C_{t-1}^2 C_{t-2}^2]. \tag{3.8c}$$

We provide expressions for  $\tilde{\lambda}_{k_{\perp}}^{(j)}(t, \phi)$  for  $j = 0, 1, \dots, 5$  in the appendix. They become cumbersome rather quickly as  $j$  increases. For  $j \geq 1$ ,  $\tilde{\lambda}_{k_{\perp}}^{(j)}(t, \phi)$  can be shown to obey the following recursion relation:

$$\tilde{\lambda}_{k_{\perp}}^{(j)}(t, \phi) = \left[ \sum_{\varepsilon=1}^j \tilde{\lambda}_{k_{\perp}}^{(j-\varepsilon)}(t, \phi) \tilde{\lambda}_{k_{\perp}}^{(\varepsilon-1)}(t-1, \phi) \right] \tilde{\lambda}_{k_{\perp}}^{(0)}(t-1, \phi). \tag{3.9}$$

Again we find it convenient to strip off the last step of the walk and study  $\hat{\lambda}_{k_{\perp}}(t, \phi)$ . After generating  $\hat{\lambda}_{k_{\perp}}^{(j)}(t, \phi)$ s from the recursion relation, we have observed that they could be expressed in the following way:

$$\hat{\lambda}_{k_{\perp}}^{(j)}(t, \phi) = \sum_{\{n_i | \sum_{i=1}^j n_i = j\}} \left[ \prod_{i=1}^{\infty} \binom{n_i + n_{i+1} - 1}{n_{i+1}} (4v^2 C_{t-i}^2)^{n_i} \right] \tag{3.10a}$$

where  $n_i = 0, 1, \dots, j$  for  $i = 1, 2, \dots, j$  and where

$$\binom{n_i + n_{i+1} - 1}{n_{i+1}} = \begin{cases} 1 & \text{if } n_i = 0 \text{ and } n_{i+1} = 0 \\ 0 & \text{if } n_i = 0 \text{ and } n_{i+1} > 0 \\ \frac{(n_i + n_{i+1} - 1)!}{(n_{i+1})!(n_i - 1)!} & \text{otherwise.} \end{cases} \tag{3.10b}$$

The restriction on the  $n_i$ 's in equation (3.10a) ensures that the contributing walks have the appropriate length. Note first of all that each cosine term appears an even number of times, making the round trip possible. Moreover, the terms are 'connected'; for instance, if a term contains a factor  $C_{t-3}$ , it must also contain factors  $C_{t-2}$  and  $C_{t-1}$ . Finally, the numerical component (the degeneracy) factorizes into terms which depend only on the number of steps in two neighbouring columns ( $n_i$  and  $n_{i+1}$ ).

Expression (3.10) may have certain practical advantages over equation (3.3) in that it selects out walks of a particular length and that it is not written in the form of a quotient. Nevertheless, we proceed now to use it in a second derivation of equation (3.3). Performing the sum of  $\hat{\lambda}_{k_{\perp}}^{(j)}(t, \phi)$  over  $j$

$$\hat{\lambda}_{k_{\perp}}(t, \phi) = \sum_{j=0}^{\infty} \hat{\lambda}_{k_{\perp}}^{(j)}(t, \phi) \tag{3.11a}$$

actually simplifies matters as it eliminates the restriction on  $n_i$ s, leading to

$$\hat{\lambda}_{k_{\perp}}(t, \phi) = \sum_{\{n_1, n_2, \dots\}} \left[ \prod_{i=1}^{\infty} \binom{n_i + n_{i+1} - 1}{n_{i+1}} (4v^2 C_{t-i}^2)^{n_i} \right]. \tag{3.11b}$$

Next carry out the sums over  $n_i$  consecutively. Using repeatedly the following sum

$$\sum_{j=0}^{\infty} \binom{j+k-1}{k} x^j = 1 + \frac{x}{(1-x)^{k+1}} \tag{3.12}$$

leads to

$$\begin{aligned} \hat{\lambda}_{k_{\perp}}(t, \phi) = & 1 + \frac{(2v)^2 C_{t-1}^2}{(1-4v^2 C_{t-1}^2)} + \frac{(2v)^4 C_{t-1}^2 C_{t-2}^2}{(1-4v^2 C_{t-1}^2)(1-4v^2 C_{t-1}^2 - 4v^2 C_{t-2}^2)} \\ & + \frac{(2v)^6 C_{t-1}^2 C_{t-2}^2 C_{t-3}^2}{(1-4v^2 C_{t-1}^2 - 4v^2 C_{t-2}^2)(1-4v^2 C_{t-1}^2 - 4v^2 C_{t-2}^2 - 4v^2 C_{t-3}^2 + 16v^4 C_{t-1}^2 C_{t-3}^2)} \\ & + O(v^8). \end{aligned} \tag{3.13}$$

Finally finding common denominators for the first two terms, then the first three terms, etc, yields expressions like

$$\begin{aligned} \hat{\lambda}_{k_{\perp}}(t, \phi) = & \frac{1-4v^2 C_{t-2}^2}{1-4v^2 C_{t-1}^2 - 4v^2 C_{t-2}^2} + O(v^6) \\ \hat{\lambda}_{k_{\perp}}(t, \phi) = & \frac{1-4v^2 C_{t-2}^2 - 4v^2 C_{t-3}^2}{1-4v^2 C_{t-1}^2 - 4v^2 C_{t-2}^2 - 4v^2 C_{t-3}^2 + 16v^4 C_{t-1}^2 C_{t-3}^2} + O(v^8) \end{aligned} \tag{3.14}$$

and so on. Hence, one arrives at

$$\hat{\lambda}_{k_{\perp}}(t, \phi) = \frac{D_{k_{\perp}}(t, \phi; 2, s)}{D_{k_{\perp}}(t, \phi; 1, s)} + O(v^{2s+2}) \tag{3.15}$$

which is identical to equation (3.3).

#### 4. Special cases

In this section, we consider the evaluation of  $\hat{\lambda}_{k_{\perp}}(t, \phi)$  in three special cases: (i) when  $v$  is small (which provides a straightforward extension of the directed-paths-only analysis in the strongly localized regime); (ii) when the magnetic flux-per-plaquette is a rational multiple of the flux quantum; and (iii) when  $\phi$  is small (weak magnetic field, for which one should be able to make some correspondence with the continuum model).

4.1. *When  $v$  is small*

The first special case we shall treat is when  $v$  is small. Under such conditions, it is reasonable to view the various quantities as power series in  $v$ . We have already seen in the previous section that

$$\hat{\lambda}_{k_{\perp}}^{(0)}(t, \phi) = 1 \tag{4.1a}$$

$$\hat{\lambda}_{k_{\perp}}^{(1)}(t, \phi) = (2v)^2 C_{t-1}^2 \tag{4.1b}$$

$$\hat{\lambda}_{k_{\perp}}^{(2)}(t, \phi) = (2v)^4 [C_{t-1}^4 + C_{t-1}^2 C_{t-2}^2] \tag{4.1c}$$

$$\hat{\lambda}_{k_{\perp}}^{(3)}(t, \phi) = (2v)^6 [C_{t-1}^6 + 2C_{t-1}^4 C_{t-2}^2 + C_{t-1}^2 C_{t-2}^4 + C_{t-1}^2 C_{t-2}^2 C_{t-3}^2] \tag{4.1d}$$

and so on. From these, we see that  $\hat{\lambda}_{k_{\perp}}(t, \phi)$  can be re-expressed as

$$\hat{\lambda}_{k_{\perp}}(t, \phi) = \exp\{(2v)^2 C_{t-1}^2\} + O(v^4). \tag{4.2a}$$

$$\hat{\lambda}_{k_{\perp}}(t, \phi) = \exp\{(2v)^2 C_{t-1}^2 + (2v)^4 [\frac{1}{2} C_{t-1}^4 + C_{t-1}^2 C_{t-2}^2]\} + O(v^6). \tag{4.2b}$$

$$\hat{\lambda}_{k_{\perp}}(t, \phi) = \exp\{(2v)^2 C_{t-1}^2 + (2v)^4 [\frac{1}{2} C_{t-1}^4 + C_{t-1}^2 C_{t-2}^2] + (2v)^6 [\frac{1}{3} C_{t-1}^6 + C_{t-1}^4 C_{t-2}^2 + C_{t-1}^2 C_{t-2}^4 + C_{t-1}^2 C_{t-2}^2 C_{t-3}^2]\} + O(v^8). \tag{4.2c}$$

Expressions (4.2a), (4.2b) and (4.2c) account accurately for backward excursions of length 2, 4, and 6, respectively.

Note that the expansion of the  $\ln[\hat{\lambda}_{k_{\perp}}(t, \phi)]$  (the argument of  $\exp\{ \}$  above) looks rather like the expansion of  $\hat{\lambda}_{k_{\perp}}(t, \phi)$  itself (equation (3.11b)) except that coefficients have an additional  $1/n_1$  factor

$$\ln[\hat{\lambda}_{k_{\perp}}(t, \phi)] = \sum_{[n_1, n_2, \dots]} \left[ \frac{1}{n_1} \prod_{i=1}^{\infty} \binom{n_i + n_{i+1} - 1}{n_{i+1}} (4v^2 C_{t-i}^2)^{n_i} \right] \tag{4.3}$$

and, of course, the  $n_1 = 0$  term is absent. We can verify this expression with repeated use of the relation

$$\sum_{j=1}^{\infty} \frac{1}{j} \binom{j+k-1}{k} x^j = \begin{cases} -\ln(1-x) & \text{if } k = 0 \\ \frac{1}{k}(1-x)^{-k} - \frac{1}{k} & \text{if } k \neq 0 \end{cases} \tag{4.4}$$

which leads to

$$\ln(\hat{\lambda}_{k_{\perp}}(t, \phi)) = \ln[D_{k_{\perp}}(t, \phi; 2, s)] - \ln[D_{k_{\perp}}(t, \phi; 1, s)] + O(v^{2r+2}) \tag{4.5}$$

which is simply the logarithm of equation (3.3a).

The form given above is particularly convenient for calculating  $G_{k_{\perp}}^{be}(0, T_0, \phi)$ , the Green function for long walks that allows backward excursions, because  $G_{k_{\perp}}^{be}(0, T_0, \phi)$  factors into  $G_{k_{\perp}}^{dir}(0, T_0, \phi)$  and a product of  $\hat{\lambda}$ 's

$$G_{k_{\perp}}^{be}(0, T_0, \phi) = \prod_{t=1}^{T_0} \hat{\lambda}_{k_{\perp}}(t, \phi) = G_{k_{\perp}}^{dir}(0, T_0, \phi) \prod_{t=1}^{T_0} \hat{\lambda}_{k_{\perp}}(t, \phi) \tag{4.6a}$$

and the exponential form converts that product into a sum:

$$G_{k_{\perp}}^{bc}(0, T_0, \phi) = G_{k_{\perp}}^{\text{dir}}(0, T_0, \phi) \exp \left\{ \sum_{i=1}^{T_0} [(2v)^2 C_{i-1}^2 + (2v)^4 [\frac{1}{2} C_{i-1}^4 + C_{i-1}^2 C_{i-2}^2] + \dots] \right\}. \tag{4.6b}$$

Let us return to the example considered in the previous section (equations (3.5) and (3.6)) in which we have examined the effect of short backward excursions (of length 2 only) on the interference pattern generated on the superlattice when the flux-per-plaquette is commensurate ( $\phi = p/q$ ). We are now in a position to extend that analysis. Substituting  $T_0 = q$  and  $\phi = p/q$  into equation (4.6b), we find

$$G_{k_{\perp}}^{bc}(0, q, p/q) = G_{k_{\perp}}^{\text{dir}}(0, q, p/q) \exp \left\{ (2v)^2 \frac{q}{2} + (2v)^4 \left[ \frac{7q}{16} + \frac{q}{8} \cos \left( \frac{2\pi p}{q} \right) \right] + \dots \right\} \tag{4.7}$$

where we have used the following relations for  $\phi = p/q$

$$\sum_{i=1}^q C_{i-1}^2 = \frac{q}{2} \quad \text{for } q \geq 2 \tag{4.8a}$$

$$\sum_{i=1}^q C_{i-1}^4 = \frac{3q}{8} \quad \text{for } q \geq 3 \tag{4.8b}$$

$$\sum_{i=1}^q C_{i-1}^2 C_{i-2}^2 = q \left[ \frac{1}{4} + \frac{1}{8} \cos \left( \frac{2\pi p}{q} \right) \right] \quad \text{for } q \geq 3. \tag{4.8c}$$

The exponential part represents the correction to the directed-paths-only approach. We see the same pattern here as before—that on the superlattice ( $T_0 = q$ ) the transverse momentum  $k_{\perp}$  does not enter the corrections (i.e. there are no structural changes) at order  $v^2$  or order  $v^4$  provided  $q \geq 3$ , suggesting that  $k_{\perp}$  enters only at order  $v^{2q}$ . Some of the sums needed to carry out this analysis to higher order are

$$\sum_{j=1}^q C_{i-j}^{2m} = \frac{q}{2^{2m}} \binom{2m}{m} \quad \text{if } q > m \tag{4.9a}$$

$$\sum_{j=i}^q C_j^{2m} C_{j+1}^{2n} = \frac{q}{2^{2m+2n-1}} \left[ \sum_{\ell=0}^n \binom{2m}{m+\ell} \binom{2n}{n-\ell} \cos \left( \frac{2\pi p \ell}{q} \right) - \frac{1}{2} \binom{2m}{m} \binom{2n}{n} \right] \tag{4.9b}$$

if  $q > m + n$ .

This last calculation is really a mixture of two special cases—small  $v$  and commensurate flux ( $\phi = p/q$ ). We will now turn our attention to the commensurate case.

#### 4.2. Commensurate flux

In the case in which the flux-per-plaquette  $\phi$  is a rational multiple ( $p/q$ ) of the flux quantum, the terms in the continued fraction expression (equation (3.2)) begin to repeat themselves after  $q$  terms. For instance, for  $\phi = \frac{1}{3}$ , one has

$$\hat{\lambda}_{k_{\perp}}(t, \frac{1}{3}) = \frac{1}{1 - \frac{4v^2 C_{t-1}^2}{1 - \frac{4v^2 C_{t-2}^2}{1 - 4v^2 C_{t-3}^2 \hat{\lambda}_{k_{\perp}}(t, \frac{1}{3})}}}. \tag{4.10a}$$

Simplifying the fraction yields

$$\hat{\lambda}_{k_{\perp}}(t, \frac{1}{3}) = \frac{1 - 4v^2 C_{t-2}^2 - 4v^2 C_{t-3}^2 \hat{\lambda}_{k_{\perp}}(t, \frac{1}{3})}{1 - 4v^2 C_{t-1}^2 - 4v^2 C_{t-2}^2 - 4v^2 C_{t-3}^2 \hat{\lambda}_{k_{\perp}}(t, \frac{1}{3})}. \quad (4.10b)$$

Performing the same procedure for any  $\phi = p/q$ , we find

$$\hat{\lambda}_{k_{\perp}}(t, p/q) = \frac{D_{k_{\perp}}(t, p/q; 2, q-1) - 4v^2 C_{t-q}^2 D_{k_{\perp}}(t, p/q, 2, q-2) \hat{\lambda}_{k_{\perp}}(t, p/q)}{D_{k_{\perp}}(t, p/q; 1, q-1) - 4v^2 C_{t-q}^2 D_{k_{\perp}}(t, p/q, 1, q-2) \hat{\lambda}_{k_{\perp}}(t, p/q)}. \quad (4.11)$$

From here we conclude that  $\hat{\lambda}_{k_{\perp}}(t, p/q)$  is the solution of the following quadratic equation:

$$A_{k_{\perp}}(t, p/q) [\hat{\lambda}_{k_{\perp}}(t, p/q)]^2 + B_{k_{\perp}}(t, p/q) \hat{\lambda}_{k_{\perp}}(t, p/q) + C_{k_{\perp}}(t, p/q) = 0 \quad (4.12a)$$

where

$$A_{k_{\perp}}(t, p/q) = 4v^2 C_{t-q}^2 D_{k_{\perp}}(t, p/q; 1, q-2) \quad (4.12b)$$

$$B_{k_{\perp}}(t, p/q) = -D_{k_{\perp}}(t, p/q; 1, q-1) - 4v^2 C_{t-q}^2 D_{k_{\perp}}(t, p/q; 2, q-2) \quad (4.12c)$$

$$C_{k_{\perp}}(t, p/q) = D_{k_{\perp}}(t, p/q; 2, q-1). \quad (4.12d)$$

The solution of the quadratic equation is:

$$\hat{\lambda}_{k_{\perp}}(t, p/q) = \frac{-B_{k_{\perp}}(t, p/q) - \sqrt{B_{k_{\perp}}^2(t, p/q) - 4A_{k_{\perp}}(t, p/q)C_{k_{\perp}}(t, p/q)}}{2A_{k_{\perp}}(t, p/q)}. \quad (4.13)$$

We have used this last expression to calculate  $\hat{\lambda}_{k_{\perp}}(t, p/q)$  for relatively small  $q$ . Some algebra and trigonometry reveal that

$$\hat{\lambda}_{k_{\perp}}(t, 1) = \frac{1 - \sqrt{1 - 16v^2 C_{t-1}^2}}{8v^2 C_{t-1}^2} \quad (4.14a)$$

$$\hat{\lambda}_{k_{\perp}}(t, \frac{1}{2}) = \frac{1 - 4v^2 + 8v^2 C_{t-2}^2 - \sqrt{(1 - 4v^2)^2 - 16v^4 \sin^2(2k_{\perp})}}{8v^2 C_{t-2}^2} \quad (4.14b)$$

$$\hat{\lambda}_{k_{\perp}}(t, \frac{1}{3}) = \frac{1 - 6v^2 + 8v^2 C_{t-3}^2 - \sqrt{(1 - 6v^2)^2 - 16v^6 \cos^2(3k_{\perp})}}{8v^2 C_{t-3}^2 (1 - 4v^2 C_{t-1}^2)}. \quad (4.14c)$$

One can see a pattern arising from these expressions. For instance, we can see here and show in general that the discriminant ( $B^2 - 4AC$ ) is independent of  $t$ .

Note that  $q = 1$  corresponds to one flux quantum per plaquette or equivalently no field at all. The small field limit, on the other hand, would require taking  $q$  to infinity. Fortunately, this limit is accessible by other means.

4.3. The small field limit

The third special case we consider here is when the magnetic field is small. Toward this end, let us perform a Taylor expansion of  $\hat{\lambda}_{k_\perp}(t, \phi)$  around  $(k_\perp - t\phi\pi)$ :

$$\hat{\lambda}_{k_\perp}(t, \phi) = \sum_{i=0}^{\infty} \Lambda_{k_\perp}^{(i)}(t, \phi) \phi^i. \tag{4.15}$$

It is our goal in this section to calculate the  $i = 0, 1$  and  $2$  terms  $\Lambda_{k_\perp}^{(i)}(t, \phi)$  above.

Instead of expanding  $\hat{\lambda}_{k_\perp}(t, \phi)$  directly, we will begin by expanding  $\hat{\lambda}_{k_\perp}^{(j)}(t, \phi)$ . Recalling equation (3.10a) for  $\hat{\lambda}_{k_\perp}^{(j)}(t, \phi)$ , we clearly need an expansion of  $C_{i-j}^\ell$  around  $(k_\perp - t\phi\pi)$

$$C_{i-j}^\ell = C_i^\ell - j\ell\phi\pi C_i^{\ell-1} S_i + \frac{1}{2}j^2\phi^2\pi^2[\ell(\ell-1)C_i^{\ell-2}S_i^2 - \ell C_i^\ell] + \dots \tag{4.16}$$

where  $S_i = \sin(k_\perp - t\phi\pi)$ . Below we present the expansions of  $\hat{\lambda}_{k_\perp}^{(j)}(t, \phi)$  to order  $\phi^2$  for  $j = 0, 1, \dots, 6$ :

$$\hat{\lambda}_{k_\perp}^{(0)}(t, \phi) = 1 \tag{4.17a}$$

$$\hat{\lambda}_{k_\perp}^{(1)}(t, \phi) = (2v)^2[C_i^2 - 2\pi\phi C_i S_i + \frac{1}{2}\pi^2\phi^2(2S_i^2 - 2C_i^2) - \dots] \tag{4.17b}$$

$$\hat{\lambda}_{k_\perp}^{(2)}(t, \phi) = (2v)^4[2C_i^4 - 10\pi\phi C_i^3 S_i + \frac{1}{2}\pi^2\phi^2(38C_i^2 S_i^2 - 14C_i^4) - \dots] \tag{4.17c}$$

$$\hat{\lambda}_{k_\perp}^{(3)}(t, \phi) = (2v)^6[5C_i^6 - 44\pi\phi C_i^5 S_i + \frac{1}{2}\pi^2\phi^2(332C_i^4 S_i^2 - 76C_i^6) - \dots] \tag{4.17d}$$

$$\hat{\lambda}_{k_\perp}^{(4)}(t, \phi) = (2v)^8[14C_i^8 - 186\pi\phi C_i^7 S_i + \frac{1}{2}\pi^2\phi^2(2246C_i^6 S_i^2 - 374C_i^8) - \dots] \tag{4.17e}$$

$$\hat{\lambda}_{k_\perp}^{(5)}(t, \phi) = (2v)^{10}[42C_i^{10} - 772\pi\phi C_i^9 S_i + \frac{1}{2}\pi^2\phi^2(13348C_i^8 S_i^2 - 1748C_i^{10}) - \dots] \tag{4.17f}$$

$$\hat{\lambda}_{k_\perp}^{(6)}(t, \phi) = (2v)^{12}[132C_i^{12} - 3172\pi\phi C_i^{11} S_i + \frac{1}{2}\pi^2\phi^2(73340C_i^{10} S_i^2 - 7916C_i^{12}) - \dots] \tag{4.17g}$$

Next, we collect terms of the same order in  $\phi$  and resum those series. Toward that end, we define

$$\Lambda_{k_\perp}^{(0)} = \sum_{j=0}^{\infty} N_j (2vC_i)^{2j} \tag{4.18a}$$

$$\Lambda_{k_\perp}^{(1)} = -\frac{2\pi S_i}{C_i} \sum_{j=0}^{\infty} P_j (2vC_i)^{2j} \tag{4.18b}$$

$$\Lambda_{k_\perp}^{(2)} = \frac{\pi^2 S_i^2}{2C_i^2} \sum_{j=0}^{\infty} Q_j (2vC_i)^{2j} - \frac{\pi^2}{2} \sum_{j=0}^{\infty} R_j (2vC_i)^{2j} \tag{4.18c}$$

From equations (4.17a-g), we can conclude, for instance, that

$$\{P_j\} = \{0, 1, 5, 22, 93, 386, 1586, \dots\}. \tag{4.19}$$

These four series are summed as follows:

$$\sum_{j=0} N_j x^j = \frac{1}{2}x[1 - (1 - 4x)^{1/2}] \quad (4.20a)$$

$$\sum_{j=0} P_j x^j = \frac{1}{2}[(1 - 4x)^{-1} - (1 - 4x)^{-1/2}] \quad (4.20b)$$

$$\sum_{j=0} Q_j x^j = \frac{5}{2}(1 - 4x)^{-5/2} - 2(1 - 4x)^{-2} - 2(1 - 4x)^{-3/2} + (1 - 4x)^{-1} + \frac{1}{2}(1 - 4x)^{-1/2} \quad (4.20c)$$

$$\sum_{j=0} R_j x^j = (1 - 4x)^{-3/2} - (1 - 4x)^{-1}. \quad (4.20d)$$

Using these we arrive at

$$\Lambda_{k_\perp}^{(0)}(t, \phi) = \frac{1 - \sqrt{1 - 16v^2 C_t^2}}{8v^2 C_t^2} \quad (4.21a)$$

$$\Lambda_{k_\perp}^{(1)}(t, \phi) = -\frac{8\pi v^2 C_t S_t}{1 - 16v^2 C_t^2} \Lambda_{k_\perp}^{(0)}(t, \phi) \quad (4.21b)$$

and

$$\Lambda_{k_\perp}^{(2)}(t, \phi) = 2\pi^2 v^2 U^{-3} \{2(S_t^2 - C_t^2) + S_t^2 U^{-2} [5 + U][1 - U^2]\} \Lambda_{k_\perp}^{(0)} \quad (4.21c)$$

where  $U = (1 - 16v^2 C_t^2)^{-1/2}$ .

Note that the linear term breaks the  $\phi \rightarrow -\phi$  symmetry. Two comments should be made: (i)  $\phi \rightarrow -\phi$  with  $x \rightarrow -x$  (or equivalently  $k_\perp \rightarrow -k_\perp$ ) is still a symmetry, and (ii) most importantly all physical properties which depend on the absolute value  $|G|$  will be invariant under  $\phi \rightarrow -\phi$  or  $x \rightarrow -x$  separately. This is consistent with the expectation that no Lorentz force (and no parity symmetry breaking) should occur for states which do not carry current density. Such effects will be seen only if the dependence of  $G$  in the  $t$  direction goes like  $e^{ik_\perp t}$  so that the states have a non-vanishing current density in the  $t$  direction [14].

## 5. The full Green function

In this section, we show how to obtain the full Green function  $G_{k_\perp}(0, T_0, \phi)$ , which is the appropriately weighted sum of all walks connecting columns 0 and  $T_0$ . (Recall that the two-dimensional problem was reduced to one-dimension.) Thus far, we have calculated  $\tilde{\lambda}_{k_\perp}(t, \phi) = G_{k_\perp}^{be}(t - 1, t, \phi)$ , which includes walks which advance from column  $t - 1$  to column  $t$  with any amount of backward excursion prior to the forward step. In this section we will find it more convenient to use the  $G_{k_\perp}^{be}(t - 1, t, \phi)$  notation. We have also already seen the product

$$G_{k_\perp}^{be}(0, T_0, \phi) = \prod_{t=0}^{T_0-1} G_{k_\perp}^{be}(t, t + 1; \phi) \quad (5.1)$$

which supplies the Green function for the restricted walks that begin at the origin and end in the  $T_0$  column, again with the last step being the only one connecting the  $T_0 - 1$  column to the  $T_0$  column. This is the Green function for the first time the walker lands in column  $T_0$ .

Now we must allow the walker to go beyond  $T_0$  or to turn back and return later to  $T_0$ . Let us denote this sum by  $\mathcal{G}_{k_\perp}^{\text{full}}(T_0, T_0; \phi)$ ; it is the Green function for all walks starting in column  $T_0$  and returning to that column. Note that  $\mathcal{G}_{k_\perp}^{\text{full}}(T_0, T_0; \phi)$  obeys the following recursion relation

$$\begin{aligned} \mathcal{G}_{k_\perp}^{\text{full}}(T_0, T_0; \phi) &= 1 + 4v^2 C_{T_0}^2 G_{k_\perp}^{\text{be}}(T_0 - 1, T_0 - 1, \phi) \mathcal{G}_{k_\perp}^{\text{full}}(T_0, T_0, \phi) \\ &\quad + 4v^2 C_{T_0+1}^2 G_{k_\perp}^{\text{fe}}(T_0 + 1, T_0 + 1, \phi) \mathcal{G}_{k_\perp}^{\text{full}}(T_0, T_0; \phi) \end{aligned} \quad (5.2a)$$

where  $G^{\text{fe}}$  is the same Green function as we have already found except that the initial position is to the right of the endpoint. Therefore, the formal solution is

$$\mathcal{G}_{k_\perp}^{\text{full}}(T_0, T_0; \phi) = [1 - 4v^2 C_{T_0}^2 G_{k_\perp}^{\text{be}}(T_0 - 1, T_0 - 1, \phi) - 4v^2 C_{T_0+1}^2 G_{k_\perp}^{\text{fe}}(T_0 + 1, T_0 + 1; \phi)]^{-1}. \quad (5.2b)$$

Finally, the full Green function  $\mathcal{G}_{k_\perp}^{\text{full}}(0, T_0; \phi)$  is simply the product

$$\mathcal{G}_{k_\perp}^{\text{full}}(0, T_0; \phi) = G_{k_\perp}^{\text{be}}(0, T_0; \phi) \mathcal{G}_{k_\perp}^{\text{full}}(T_0, T_0; \phi), \quad (5.3a)$$

$$\mathcal{G}_{k_\perp}^{\text{full}}(0, T_0, \phi) = \frac{\prod_{t=0}^{T_0-1} G_{k_\perp}^{\text{be}}(t, t+1; \phi)}{[1 - 4v^2 C_{T_0}^2 G_{k_\perp}^{\text{be}}(T_0 - 1, T_0 - 1; \phi) - 4v^2 C_{T_0+1}^2 G_{k_\perp}^{\text{fe}}(T_0 + 1, T_0 + 1; \phi)]}. \quad (5.3b)$$

This last equation is the central equation of this paper: it gives the formal expression for the Green function of an electron on a lattice in a magnetic field. The dependence on  $x$  can be obtained by an inverse Fourier transform with respect to  $\exp(ik_\perp x)$ . In principle, all physical properties may be obtained from this expression. This is certainly true for  $E$  outside the spectrum. For the energies within the 'butterfly', one may still use this formula for walks which do not reach the boundaries of the system (or wrap around the torus, for if periodic boundary conditions are assumed, other delicate questions then arise [13] if the two sizes  $L_\perp$  and  $L_\parallel$  are not integer multiples of  $q$ ).

In the next section, we look at the Green function for charged particles propagating within the staggered-flux configurations that have been studied in the context of high- $T_c$  superconductivity.

## 6. Staggered flux configurations

Beginning with the work of Affleck and Marston [11], the notion of staggered flux configurations was introduced and studied in the context of theories of high- $T_c$  superconductivity. It may, therefore, be useful to have expressions for electron propagation in such flux configurations. On a square lattice, such a configuration will be given by a chessboard arrangement of interpenetrating sets of plaquettes with (say)  $+\phi$  on the white squares and  $-\phi$  on the black ones (see figure 2). The most studied case is  $\phi = \pi$ , for which the phase factors acquired around the plaquettes are  $\pm 1$ , so that time reversal symmetry is



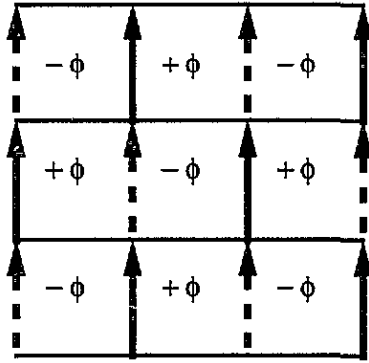


Figure 2. The staggered flux configuration.

unbroken. The sum of walks in this particular case was obtained by Khveshchenko *et al* [15]. In the present section, we obtain these sums for arbitrary staggered flux  $\pm\phi$ .

First, we must choose a gauge: this time we assign the phases only to the vertical bonds of a square lattice. Considering them as directed upwards, we assign a factor  $\gamma = e^{i\phi/2}$  for bonds pointing from sublattice A to sublattice B (shown as a broken line in figure 2) and  $\gamma^* = e^{-i\phi/2}$  to the bonds pointing from sublattice B to A (full line in figure 2). So a staggered arrangement of phases  $\pm\phi/2$  is assigned to the (upward-pointing) vertical bonds.

Next, we assign a factor  $\alpha$  ( $\alpha^{-1}$ ) to a step in the positive (negative)  $x$  direction and similarly  $\beta$  ( $\beta^{-1}$ ) for the  $y$  direction. A natural choice is  $\alpha = \exp(ik_x)$  and  $\beta = \exp(ik_y)$  which yields the Fourier transform of the generating functions. We define the following generating functions: first, the local generating functions for a step

$$\begin{aligned} A \rightarrow B : \quad & g_A = \alpha + \alpha^{-1} + \gamma(\beta + \beta^{-1}) \\ B \rightarrow A : \quad & g_B = \alpha + \alpha^{-1} + \gamma^{-1}(\beta + \beta^{-1}) \end{aligned} \tag{6.1}$$

one for each sublattice; then a global generating function for all paths of length  $2L$ :

$$G_{2L}^{\text{stag}}(\phi) = g_A^L g_B^L = [4(\cos^2 k_x + \cos^2 k_y + 2 \cos(\phi/2) \cos k_x \cos k_y)]^L. \tag{6.2}$$

The probability amplitude to reach a given site  $(x, y)$  is the coefficient of  $\alpha^x \beta^y$  (up to a trivial normalization) of the inverse Fourier transform:

$$P_{2L}^{\text{stag}}(\phi) \propto \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dk_x dk_y G_{2L}^{\text{stag}}(\phi) e^{-ik_x x - ik_y y} \tag{6.3}$$

where the system size is assumed to be infinite and we have thus gone over to integrals over the momenta. Note that in order to obtain the normalized Green function,  $g_A$  and  $g_B$  must each be multiplied by  $\frac{1}{2}$  and  $G_{2L}$  by  $2^{-2L}$ .

Of particular interest are the expressions for walks which return to their initial point. The weighted closed magnetic walks are given by the coefficient of  $\alpha^0 \beta^0 = 1$  in the product:

$$G_{2L}^{\text{stag}}(\phi) = [(\alpha + \alpha^{-1}) + \gamma(\beta + \beta^{-1})]^L [(\alpha + \alpha^{-1}) + \gamma^{-1}(\beta + \beta^{-1})]^L \tag{6.4}$$

where  $\gamma = e^{i\phi/2}$ . Toward this end, we expand the above products as follows

$$[(\alpha + \alpha^{-1}) + \gamma(\beta + \beta^{-1})]^L = \sum_{L_x^{(A)}} \binom{L}{L_x^{(A)}} (\alpha + \alpha^{-1})^{L_x^{(A)}} [\gamma(\beta + \beta^{-1})]^{L_y^{(A)}} \tag{6.5a}$$

$$[(\alpha + \alpha^{-1}) + \gamma^{-1}(\beta + \beta^{-1})]^L = \sum_{L_x^{(B)}} \binom{L}{L_x^{(B)}} (\alpha + \alpha^{-1})^{L_x^{(B)}} [\gamma^{-1}(\beta + \beta^{-1})]^{L_y^{(B)}} \tag{6.5b}$$

where  $L_x^{(A)}$  ( $L_y^{(A)}$ ) is the number of horizontal (vertical) steps initiated from the A sublattice and likewise for the B sublattice. Consequently,  $L_x^{(A)} + L_y^{(A)} = L_x^{(B)} + L_y^{(B)} = L$ . Next, we expand the remaining  $\alpha$  products, introducing the variables  $\ell_{+x}^{(A)}$  and  $\ell_{+x}^{(B)}$  which represent the number of steps in the positive  $x$  direction initiated on the A and B sublattices, respectively. The coefficients of  $\alpha$  will thus be

$$\sum_{\ell_{+x}^{(A)}}^{L_x^{(A)}} \sum_{\ell_{+x}^{(B)}}^{L_x^{(B)}} \binom{L_x^{(A)}}{\ell_{+x}^{(A)}} \alpha^{2\ell_{+x}^{(A)} - L_x^{(A)}} \binom{L_x^{(B)}}{\ell_{+x}^{(B)}} \alpha^{2\ell_{+x}^{(B)} - L_x^{(B)}}. \quad (6.6)$$

To obtain the  $\alpha^0$  term, the following relation must hold

$$2(\ell_{+x}^{(A)} + \ell_{+x}^{(B)}) = L_x^{(A)} + L_x^{(B)} = L_x \quad (6.7)$$

where  $L_x$  is the total number of horizontal steps. Note that  $L_x$  must be even. Replacing  $\ell_{+x}^{(B)}$  by  $\frac{1}{2}L_x - \ell_{+x}^{(A)}$ , we find that the coefficient of  $\alpha^0$  is

$$\sum_{\ell_{+x}^{(A)}}^{L_x^{(A)}} \binom{L_x^{(A)}}{\ell_{+x}^{(A)}} \binom{L_x^{(B)}}{\frac{1}{2}L_x - \ell_{+x}^{(A)}} = \binom{L_x}{\frac{1}{2}L_x}. \quad (6.8)$$

Repeating the same procedure for the  $y$  direction and combining the results yields

$$G_{2L}^{\text{stag}}(\phi) = \sum_{L_x^{(A)}}^L \sum_{L_x^{(B)}}^L \binom{L}{L_x^{(A)}} \binom{L_x}{\frac{1}{2}L_x} \binom{L}{L_x^{(B)}} \binom{L_y}{\frac{1}{2}L_y} \gamma^{L_y^{(A)} - L_y^{(B)}} \quad (6.9a)$$

$$\times \sum_{L_x=0}^L \sum_{L_x^{(A)}=0}^{L_x} \binom{L}{L_x^{(A)}} \binom{L_x}{\frac{1}{2}L_x} \binom{L}{L_x - L_x^{(A)}} \binom{L_y}{\frac{1}{2}L_y} \gamma^{2L_y^{(A)} - L_y} \quad (6.9b)$$

$$= \sum_{L_x} \binom{L_x}{\frac{1}{2}L_x} \binom{2L - L_x}{L - \frac{1}{2}L_x} \sum_{L_x^{(A)}} \binom{L}{L_x^{(A)}} \binom{L}{L_x - L_x^{(A)}} \gamma^{L_x - 2L_x^{(A)}} \quad (6.9c)$$

where we have used  $L_y^{(A)} = L - L_x^{(A)}$  and  $L_y = 2L - L_x$ .

Using equation (6.9) we have found  $G_{2L}^{\text{stag}}$  for several  $L$ 's. They are:

$$G_0^{\text{stag}} = 1 \quad (6.10a)$$

$$G_2^{\text{stag}} = 4 \quad (6.10b)$$

$$G_4^{\text{stag}} = 28 + 8 \cos \phi \quad (6.10c)$$

$$G_6^{\text{stag}} = 256 + 144 \cos \phi \quad (6.10d)$$

$$G_8^{\text{stag}} = 2716 + 2112 \cos \phi + 72 \cos 2\phi \quad (6.10e)$$

$$G_{10}^{\text{stag}} = 31\,504 + 29\,600 \cos \phi + 2400 \cos 2\phi \quad (6.10f)$$

$$G_{12}^{\text{stag}} = 387\,136 + 411\,840 \cos \phi + 54\,000 \cos 2\phi + 800 \cos 3\phi \quad (6.10g)$$

$$G_{14}^{\text{stag}} = 4\,951\,552 + 5\,752\,992 \cos \phi + 1\,034\,880 \cos 2\phi + 39\,200 \cos 3\phi \quad (6.10h)$$

$$G_{16}^{\text{stag}} = 65\,218\,204 + 80\,950\,016 \cos \phi + 18\,267\,200 \cos 2\phi + 1\,191\,680 \cos 3\phi \\ + 9800 \cos 4\phi \quad (6.10i)$$

$$G_{18}^{\text{stag}} = 878\,536\,624 + 1\,148\,084\,928 \cos \phi + 307\,577\,088 \cos 2\phi + 29\,070\,720 \cos 3\phi \\ + 635\,040 \cos 4\phi \quad (6.10j)$$

$$G_{20}^{\text{stag}} = 12\,046\,924\,528 + 16\,407\,496\,800 \cos \phi + 5\,030\,575\,200 \cos 2\phi + 625\,312\,800 \cos 3\phi \\ + 24\,343\,200 \cos 4\phi + 127\,008 \cos 5\phi. \quad (6.10k)$$

$G_{2L}^{\text{stag}}$  can also be obtained as the coefficient of  $v^{2L}$  in the following expression:

$$\frac{1}{2\pi} \int_0^{2\pi} dk_{\perp} \{1 - 8v^2 - 8v^2 \cos(\phi/2) \cos(2k_{\perp}) + 16v^4 \sin^2(\phi/2) \sin^2(2k_{\perp})\}^{-1/2}. \quad (6.10)$$

As we work toward procuring a more general expression, let us concentrate on the following sum:

$$\sum_{L_x^{(A)}}^{L_x} \binom{L}{L_x^{(A)}} \binom{L}{L_x - L_x^{(A)}} \gamma^{L_x - 2L_x^{(A)}} \quad (6.11a)$$

which is the latter portion of equation (6.9c). Introducing  $L_x^{(B)} = L_x - L_x^{(A)}$  this may be written as

$$\begin{aligned} & \sum_{L_x^{(A)}}^L \sum_{L_x^{(B)}}^L \binom{L}{L_x^{(A)}} \binom{L}{L_x^{(B)}} \gamma^{L_x^{(B)} - L_x^{(A)}} \delta[L_x - L_x^{(A)} - L_x^{(B)}] \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \sum_{L_x^{(A)}}^L \sum_{L_x^{(B)}}^L \binom{L}{L_x^{(A)}} \binom{L}{L_x^{(B)}} \gamma^{L_x^{(B)} - L_x^{(A)}} e^{-i\theta[L_x - L_x^{(A)} - L_x^{(B)}]} \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\theta [1 + \gamma e^{i\theta}]^L [1 + \gamma^{-1} e^{i\theta}]^L e^{-i\theta L_x} \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{i\theta(L-L_x)} [2 \cos \theta + (\gamma + \gamma^{-1})]^L \\ &= \frac{2^L}{2\pi} \int_0^{2\pi} d\theta e^{i\theta(L-L_x)} [\cos \theta + \cos \frac{1}{2}\phi]^L. \end{aligned} \quad (6.11b)$$

We may now use the identity

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta [\mu + \sqrt{\mu^2 - 1} \cos \theta]^n \begin{pmatrix} \cos m\theta \\ \sin m\theta \end{pmatrix} = \frac{n!}{(n+m)!} \begin{pmatrix} P_n^m(\mu) \\ 0 \end{pmatrix}. \quad (6.12)$$

We identify  $\cos \frac{1}{2}\phi = \mu / \sqrt{\mu^2 - 1}$  which implies  $\mu = \pm i \cot \frac{1}{2}\phi$  such that the integral becomes

$$\frac{2^L}{2\pi} \frac{1}{(\mu^2 - 1)^{L/2}} \int_0^{2\pi} d\theta e^{i\theta(L-L_x)} [\mu + \sqrt{\mu^2 - 1} \cos \theta]^L. \quad (6.13)$$

Therefore for the sum

$$\sum_{L_x^{(A)}}^{L_x} \binom{L}{L_x^{(A)}} \binom{L}{L_x - L_x^{(A)}} \gamma^{L_x - 2L_x^{(A)}} = (-i)^{L/2} \left(\sin \frac{\phi}{2}\right)^{L/2} \frac{2^L L!}{(2L - L_x)!} P_L^{(L-L_x)} \left(i \cot \frac{\phi}{2}\right). \quad (6.14)$$

The final result is

$$G_{2L}^{\text{stag}} = C(L) \sum_{L_x=0}^{2L} \binom{L_x}{L_x/2} \binom{2L - L_x}{L - L_x/2} \frac{1}{(2L - L_x)!} P_L^{(L-L_x)} \left(i \cot \frac{\phi}{2}\right) \quad (6.15a)$$

with

$$C(L) = (-i)^{L/2} \left( \sin \frac{\phi}{2} \right)^{L/2} 2^L L! \tag{6.15b}$$

We find agreement with the known cases [15] of  $\phi = 0$  for which equation (6.14) yields  $\binom{2L}{L_x}$  and  $\phi = \pi$  for which it gives  $\binom{L}{L_x/2}$ .

One can generalize the expressions to walks that do not close. To reach a given point  $(x_0, y_0)$  the only difference will be in the summation over  $l_{+x}^{(A)}$  since what we need now is the coefficient of  $\alpha^{x_0}$  or

$$2(l_{+x}^{(A)} + l_{+x}^{(B)}) - (L_x^{(A)} + L_x^{(B)}) = x_0. \tag{6.16}$$

Therefore  $l_{+x}^{(B)} = \frac{1}{2}L_x + \frac{1}{2}x_0 - l_{+x}^{(A)}$  and the sum is

$$\sum_{l_{+x}^{(A)}=0}^{L_x^{(A)}} \binom{L_x^{(A)}}{l_{+x}^{(A)}} \binom{L_x^{(B)}}{L_x/2 + x_0/2 - l_{+x}^{(A)}} = \binom{L_x}{L_x/2 + x_0/2}. \tag{6.17}$$

Likewise (to get the coefficient of  $\beta^{y_0}$ ) the summation in the  $y$  direction yields

$$\gamma^{L_y^{(A)} - L_y^{(B)}} \binom{L_y}{L_y/2 + y_0/2} \tag{6.18}$$

and the final expression changes to

$$G_{2L}^{\text{stag}} = C(L) \sum_{L_x=0}^{2L} \binom{L_x}{L_x/2 + x_0/2} \binom{2L - L_x}{L - L_x/2 + y_0/2} \frac{1}{(2L - L_x)!} P_L^{(L-L_x)} \left( i \cot \frac{\phi}{2} \right). \tag{6.19}$$

Alternatively, one could choose  $\alpha = e^{ik_x}$  and  $\beta = e^{ik_y}$  and study  $G_{2L}(\alpha, \beta)$  which can then be Fourier transformed back into real space.

### 7. Discussion

In this paper we have derived expressions for the square-lattice Green function of a charged particle in a magnetic field. We have also obtained similar expressions for the so-called staggered flux configurations with arbitrary flux  $\pm\phi$ . The results were obtained within the Peierls ansatz in which the magnetic field is represented by an extra phase acquired by the nearest-neighbour hopping term.

Our starting point was the directed-paths-only approach initiated in the study of strongly localized wavefunctions. From there, we have shown how to include systematically the returning loops and obtain the Green function for walks with backward excursions, and then we have demonstrated how the full Green functions were realized in terms of the former.

Our main goal was to see if the inclusion of small returning loops would modify in some essential way the behaviour found in the directed-paths approach. The analysis of equations (3.5), (3.6) and (4.1)–(4.8) shows that the answer is negative in the sense that

the product over cosines with changing phases obtained for directed paths is modified into a product of more complicated trigonometric expressions containing the same cosines. Of course, the behaviour will change but the essential features: strong sensitivity to the value of the field, simple scaling in  $q$  for the commensurate case  $\phi = p/q$ , and an aperiodic but deterministic behaviour for irrational  $\phi$ , all are going to survive the addition of the small returning loops. Repeating the analysis done in [10] for the product of the new trigonometric expressions would be a formidable challenge, since even for the former case, the analysis was a substantial and sophisticated task.

Our expressions might aid in the study of other interesting issues as well: for example, the transmission through a slab of finite width, in which the wavefunction does not decay all the way to zero, might also be studied. An especially interesting question concerns the emergence of a Lorentz force (and the associated parity symmetry breaking) in the case which deals with such a current carrying state.

Another interesting issue is how the continuum limit may be approached: naively, one would expect that if the magnetic flux is small such that the magnetic length is much longer than the lattice spacing (but smaller than the system size) the lattice results will coincide with those obtained in the continuum. In particular, the exponential decay should change into a Gaussian  $\sim \exp\{-cBr^2\}$  behaviour [4].

Finally, these Green functions should be correct (analytically) for energies within the band. One should, of course, add a small imaginary part to the energy (and define advanced and retarded Green functions) to deal with the singular behaviour at the eigenenergies. In addition, the question of boundary conditions for walks hitting the boundaries should be handled with special care. In principle at least, all physical information could be extracted from these Green functions. It will certainly not be straightforward to extract physically relevant information from the formal expressions, but in view of the importance of such information for many physical problems of interest (especially the many-body extensions studied in the context of the fractional quantum Hall effect, high- $T_c$  superconductivity, anyonic physics, etc), it may be worthwhile to pursue those directions.

### Acknowledgment

Acknowledgment is made to the donors of The Petroleum Research Fund, administered by the ACS, for support of this research.

### Appendix

In section 3 we have introduced the function  $\bar{\lambda}_{k_\perp}(t, \phi)$ , the eigenvalue of the transfer matrix  $\bar{T}(t, \phi)$  which allows any amount of 'backward excursion' followed by a single forward step. We called  $\bar{\lambda}_{k_\perp}^{(j)}(t, \phi)$  the contribution with path length  $(2j + 1)$ . Using the recursion relation (3.9), we have generated  $\bar{\lambda}_{k_\perp}^{(j)}(t, \phi)$  for  $j = 0, 1, \dots, 5$ . They are:

$$\bar{\lambda}_{k_\perp}^{(0)}(t, \phi) = (2v)C_t \tag{A.1a}$$

$$\bar{\lambda}_{k_\perp}^{(1)}(t, \phi) = (2v)^3 C_t C_{t-1}^2 \tag{A.1b}$$

$$\bar{\lambda}_{k_\perp}^{(2)}(t, \phi) = (2v)^5 C_t [C_{t-1}^4 + C_{t-1}^2 C_{t-2}^2] \tag{A.1c}$$

$$\bar{\lambda}_{k_\perp}^{(3)}(t, \phi) = (2v)^7 C_t [C_{t-1}^6 + 2C_{t-1}^4 C_2^2 + C_{t-1}^2 C_{t-2}^4 + C_{t-1}^2 C_{t-2}^2 C_{t-3}^2] \tag{A.1d}$$

$$\begin{aligned} \tilde{\lambda}_{k_{\perp}}^{(4)}(x, \phi) = (2v)^9 C_t [ & C_{t-1}^8 + 3C_{t-1}^6 C_{t-2}^2 + 3C_{t-1}^4 C_{t-2}^4 + 2C_{t-1}^4 C_{t-2}^2 C_{t-3}^2 + C_{t-1}^2 C_{t-2}^6 \\ & + 2C_{t-1}^2 C_{t-2}^4 C_{t-3}^2 + C_{t-1}^2 C_{t-2}^2 C_{t-3}^4 + C_{t-1}^2 C_{t-2}^2 C_{t-3}^2 C_{t-4}^2 ] \end{aligned} \quad (\text{A.1e})$$

$$\begin{aligned} \tilde{\lambda}_{k_{\perp}}^{(5)}(t, \phi) = (2v)^{11} C_t [ & C_{t-1}^{10} + 4C_{t-1}^8 C_{t-2}^2 + 6C_{t-1}^6 C_{t-2}^4 + 3C_{t-1}^6 C_{t-2}^2 C_{t-3}^2 + 4C_{t-1}^4 C_{t-2}^6 \\ & + 6C_{t-1}^4 C_{t-2}^4 C_{t-3}^2 + 2C_{t-1}^4 C_{t-2}^2 C_{t-3}^4 + 2C_{t-1}^4 C_{t-2}^2 C_{t-3}^2 C_{t-4}^2 + C_{t-1}^2 C_{t-2}^8 \\ & + 3C_{t-1}^2 C_{t-2}^6 C_{t-3}^2 + 3C_{t-1}^2 C_{t-2}^4 C_{t-3}^4 + 2C_{t-1}^2 C_{t-2}^4 C_{t-3}^2 C_{t-4}^2 + C_{t-1}^2 C_{t-2}^2 C_{t-3}^6 \\ & + 2C_{t-1}^2 C_{t-2}^2 C_{t-3}^4 C_{t-4}^2 + C_{t-1}^2 C_{t-2}^2 C_{t-3}^2 C_{t-4}^4 + C_{t-1}^2 C_{t-2}^2 C_{t-3}^2 C_{t-4}^2 C_{t-5}^2 ] \end{aligned} \quad (\text{A.1f})$$

where  $C_t = \cos(k_{\perp} - t\phi\pi)$ . These expressions can be seen to obey equation (3.10a).

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